Arguments against local hidden variable theories including a proof of the CHSH inequality

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Introduction

This is optional, non-assessable material, which extends the material covered in Chapter 6 of Book 2.

If you are intrigued or puzzled by hidden variables and Bohm-type experiments, these notes should give you deeper insights and understanding. They unpack what local hidden variable theories have to say about Bohm-type experiments and include a proof of the CHSH inequality, thereby completing the analysis needed to undermine local hidden-variable theories.

Section 1.1 reminds you about Bohm-type experiments using pairs of spin-$\frac{1}{2}$ particles. It also reviews the predictions of quantum mechanics for the probabilities of various spin measurements taken in different directions. This is not new material, but is included here for convenience. Section 1.2 reviews the basic assumptions behind local hidden variable theories. Section 1.3 then takes a specific example and tries to use hidden variables to reproduce the quantum-mechanical probabilities. Our analysis shows that this cannot be done: local hidden variable theories cannot reproduce all the results of quantum mechanics (Bell's theorem).

A more general approach uses the CHSH inequality, which is one version of Bell's inequalities. Section 2.1 introduces the correlation functions needed to form the CHSH inequality and obtains the quantum-mechanical prediction for the CHSH sum, Σ. Section 2.2 shows that local hidden variable theories impose a restriction on Σ. This restriction is known as the CHSH inequality and is proved here. The fact that quantum mechanics violates the CHSH inequality is another example of Bell's theorem. Finally, Section 2.3 summarizes why local hidden variable theories are untenable. An appendix briefly discusses extensions of the material discussed here.

1. A Bohm-type experiment

1.1 Quantum mechanical predictions

We consider a Bohm-type experiment of the kind described in Section 6.3 of Book 2. Pairs of spin-$\frac{1}{2}$ particles emerge from a source in the entangled spin state

$$|\text{singlet}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{1.1}$$

The two particles move in opposite directions along the y-axis until they are very far apart (Figure 1.1).

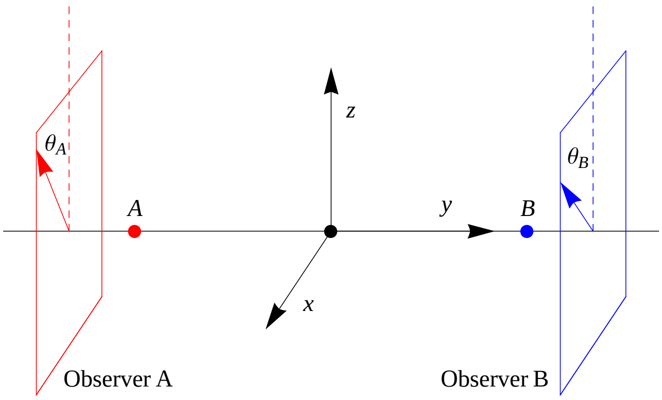

Figure 1.1 Axes chosen for spin measurements by Observer A (red) and by Observer B (blue).

Observer A measures the spin component of particle A along an axis in the xz-plane in the direction defined by the angle θ_A. Similarly, Observer B measures the spin component of particle B along an axis in the xz-plane in the direction defined by the angle θ_B. In each case, the result obtained is either spin-up or spin-down (i.e. $+\hbar/2$ or $-\hbar/2$). The two measurements are taken practically simultaneously so that no message travelling at or below the speed of light could propagate from one measurement event to influence the other measurement event.

Quantum mechanics allows us to calculate the probability P_uu that both observers will obtain spin-up results relative to their chosen axes, and also the probability P_dd that both observers will obtain spin-down results relative to these axes. The calculations are explained in Book 2 Section 6.3 but are briefly repeated here for ease of reference.

Consider a measurement axis in the xz-plane, defined by the angle θ. For brevity, we will refer to this as the θ-axis and denote the spin-up and spin-down kets defined relative to it by $|\uparrow_\theta\rangle$ and $|\downarrow_\theta\rangle$. Then putting $\phi = 0$ in Equation 3.24 of Book 2 gives

$$|\uparrow_\theta\rangle = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{bmatrix} \quad \text{and} \quad |\downarrow_\theta\rangle = \begin{bmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}, \tag{1.2}$$

so we have

$$\langle \uparrow_\theta \mid \uparrow \rangle = \cos(\theta/2) \quad \text{and} \quad \langle \uparrow_\theta \mid \downarrow \rangle = \sin(\theta/2) \tag{1.3a}$$

$$\langle \downarrow_\theta \mid \uparrow \rangle = -\sin(\theta/2) \quad \text{and} \quad \langle \downarrow_\theta \mid \uparrow \rangle = \cos(\theta/2). \tag{1.3b}$$

In the singlet state, the probability for getting spin-up for particle A along the θ_A-axis *and* spin-up for particle B along the θ_B-axis is

$$\begin{aligned} P_\mathrm{uu} &\equiv \Big|\frac{1}{\sqrt{2}}\langle \uparrow_{\theta_\mathrm{A}} \uparrow_{\theta_\mathrm{B}} \mid \uparrow\downarrow - \downarrow\uparrow \rangle\Big|^2 \\ &= \frac{1}{2}\Big|\langle \uparrow_{\theta_\mathrm{A}} \mid \uparrow \rangle\langle \uparrow_{\theta_\mathrm{B}} \mid \downarrow \rangle - \langle \uparrow_{\theta_\mathrm{A}} \mid \downarrow \rangle\langle \uparrow_{\theta_\mathrm{B}} \mid \uparrow \rangle\Big|^2. \end{aligned}$$

Using Equation 1.3a, we obtain

$$\begin{aligned} P_\mathrm{uu} &= \frac{1}{2}\Big|\cos(\theta_\mathrm{A}/2)\sin(\theta_\mathrm{B}/2) - \sin(\theta_\mathrm{A}/2)\cos(\theta_\mathrm{B}/2)\Big|^2 \\ &= \frac{1}{2}\Big|\sin(\theta_\mathrm{B}/2)\cos(\theta_\mathrm{A}/2) - \cos(\theta_\mathrm{B}/2)\sin(\theta_\mathrm{A}/2)\Big|^2 \\ &= \frac{1}{2}\sin^2[(\theta_\mathrm{B} - \theta_\mathrm{A})/2]. \end{aligned} \tag{1.4}$$

Remember: $\sin(B - A) = \sin B \cos A - \cos B \sin A$

A similar calculation gives the probability of getting two spin-down results. In this case,

$$\begin{aligned} P_\mathrm{dd} &= \Big|\frac{1}{\sqrt{2}}\langle \downarrow_{\theta_\mathrm{A}} \downarrow_{\theta_\mathrm{B}} \mid \uparrow\downarrow - \downarrow\uparrow \rangle\Big|^2 \\ &= \frac{1}{2}\Big|\langle \downarrow_{\theta_\mathrm{A}} \mid \uparrow \rangle\langle \downarrow_{\theta_\mathrm{B}} \mid \downarrow \rangle - \langle \downarrow_{\theta_\mathrm{A}} \mid \downarrow \rangle\langle \downarrow_{\theta_\mathrm{B}} \mid \uparrow \rangle\Big|^2. \end{aligned}$$

Using Equation 1.3b, we obtain

$$\begin{aligned} P_\mathrm{dd} &= \frac{1}{2}\Big| - \sin(\theta_\mathrm{A}/2)\cos(\theta_\mathrm{B}/2) - \cos(\theta_\mathrm{A}/2)(-\sin(\theta_\mathrm{B}/2))\Big|^2 \\ &= \frac{1}{2}\Big|\sin(\theta_\mathrm{B}/2)\cos(\theta_\mathrm{A}/2) - \cos(\theta_\mathrm{B}/2)\sin(\theta_\mathrm{A}/2)\Big|^2 \\ &= \frac{1}{2}\sin^2[(\theta_\mathrm{B} - \theta_\mathrm{A})/2]. \end{aligned} \tag{1.5}$$

Hence the probability that both observers will *agree* about their results (both getting spin-up or both getting spin down) is

$$P_\mathrm{agree} = P_\mathrm{uu} + P_\mathrm{dd} = \sin^2[(\theta_\mathrm{B} - \theta_\mathrm{A})/2]. \tag{1.6}$$

A particular choice of angles

We now specialize to the case considered in Exercise 6.6 of Book 2 where, for each pair of particles, Observer A has two choices for θ_A and Observer B has two choices for θ_B.

The choices are:

$$\theta_\mathrm{A} = 0^\circ \equiv \alpha_1 \quad \text{or} \quad \theta_\mathrm{A} = 90^\circ \equiv \alpha_2$$

$$\theta_\mathrm{B} = 45^\circ \equiv \beta_1 \quad \text{or} \quad \theta_\mathrm{B} = -45^\circ \equiv \beta_2,$$

and these are illustrated in Figure 1.2.

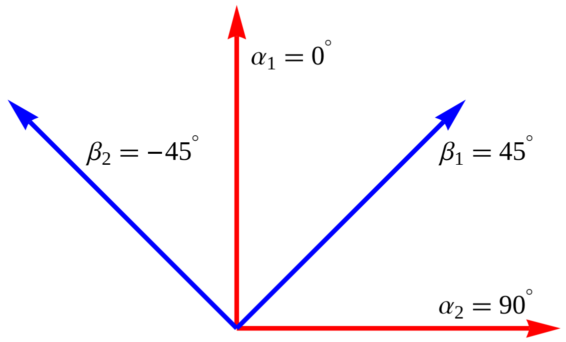

Figure 1.2 Allowed axes for the spin measurement of Particle A (α_1 and α_2) and for the spin measurement of Particle B (β_1 and β_2).

When a pair of particles is emitted from the source each observer chooses randomly, with equal probabilities, one of the two axes available to him, and a series of results is collected for many pairs of particles. Taking both observers into account, we have four different *axis combinations* which we shall write as (α_1, β_1), (α_1, β_2), (α_2, β_1) and (α_2, β_2). These four combinations are shown in Figure 1.3

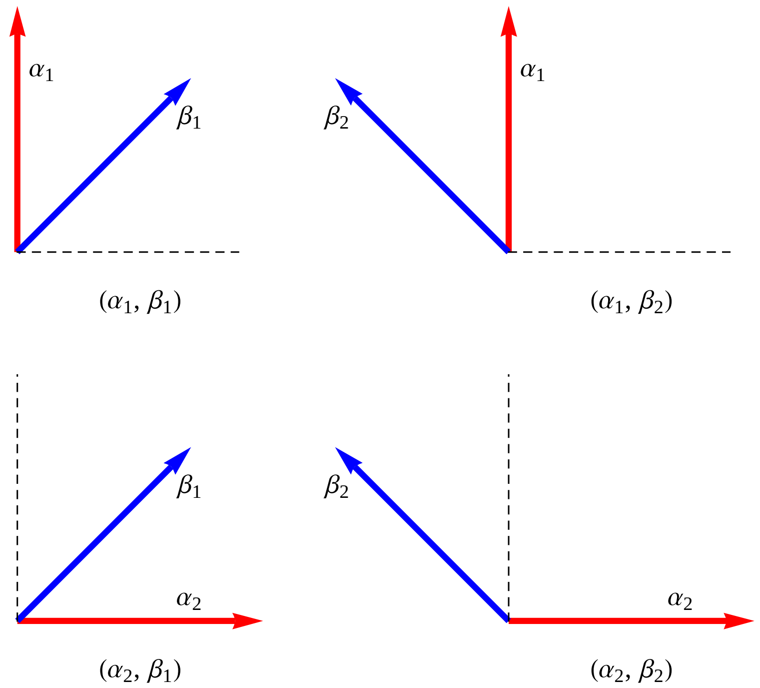

Figure 1.3 The four axis combinations used in the Bohm-type experiment considered here. Axis α_1 or α_2 is used to measure the spin of Particle A while Axis β_1 or β_2 is used to measure the spin of Particle B.

In a long run of measurements, each of the four axis combinations occurs about a quarter of the time. For each combination, we can calculate the quantum mechanical probability that both observers will *agree* about their results (both getting spin-up or both getting spin down).

Using Equation 1.6, we find that:

$$\begin{aligned}
&\text{For } (\alpha_1, \beta_1): \qquad P_{\text{agree}} = \sin^2(-22.5^\circ) = 0.146 \\
&\text{For } (\alpha_1, \beta_2): \qquad P_{\text{agree}} = \sin^2(+22.5^\circ) = 0.146 \\
&\text{For } (\alpha_2, \beta_1): \qquad P_{\text{agree}} = \sin^2(-22.5^\circ) = 0.146 \\
&\text{For } (\alpha_2, \beta_2): \qquad P_{\text{agree}} = \sin^2(+67.5^\circ) = 0.854.
\end{aligned} \tag{1.7}$$

The most striking feature of these predictions is that the two observers agree quite rarely in the first three cases, but they agree very often in the last case. These are the predictions of quantum mechanics. They are not too surprising since the two spins in a singlet state give opposite results along any single axis. It is therefore reasonable to expect that axes separated by an acute angle ($< 90^\circ$) will give less agreement than axes separated by an obtuse angle ($> 90^\circ$).

1.2 The hidden variables description

We shall now try to reproduce the quantum predictions of Equation 1.7 using a local hidden variable theory that attempts to extend classical ideas into the quantum domain. The hidden variable theory will not be based on any deep microscopic insights: it will simply be rigged up in such a way that it has a fighting chance of reproducing the results of quantum mechanics. To follow the argument, you will have to put your knowledge of quantum mechanics on one side. Here, we are arguing classically, as best we can.

We adopt the principle of **realism** which asserts that quantities have values independently of any measurement. In the context of hidden variables, it means that quantities that are regarded as being probabilistic in quantum mechanics are assumed to be completely determined by the values of certain hidden variables. If we knew the values of all these hidden variables, we would be able to predict the results of all measurements taken on the system. According to this view, the apparent indeterminacy of quantum mechanics arises from our ignorance of the hidden variables.

For the Bohm-type experiment described earlier, we shall suppose that each particle has definite values of the hidden variables that determine the values of all spin components. For example, if the hidden variable controlling the result of a spin measurement along the x-axis has a certain value (or lies in a certain range) we get the value $S_x = +\hbar/2$ with certainty; if it has an alternative value (or lies in an alternative range), we get $S_x = -\hbar/2$, also with certainty. To get agreement with the experimental observation of spin quantization, we assume that these are the only possible values allowed for S_x. The values of other hidden variables determine the results of spin measurements taken in other directions, which also have the possible values $+\hbar/2$ or $-\hbar/2$.

The hidden variable model assumes that each pair of particles emerges from the source with definite values for all hidden variables. In the course of an experiment, many pairs of particles are emitted. The various possible values of the hidden variables are assumed to occur at random

within a probability distribution that exactly mimics the results of measurements predicted by quantum mechanics. For each pair of particles, there may be built-in correlations. For example, in the case described as a singlet state in quantum mechanics, the values of the hidden variables controlling the spin results of one particle along a given axis will be the opposite of those for the other particle along that axis. This guarantees that the two particles will give opposite results when their spin components are measured along the same axis. Such correlations would emerge naturally from a conservation law; if the initial state had no angular momentum about an origin at the source, the total angular momentum of the two emerging particles would also be equal to zero, consistent with opposite spins.

By the time the two particles have moved far apart, any information about the spin measurement on particle A will travel too slowly to influence the result of the spin measurement on particle B (assuming that information can travel no faster than the speed of light in a vacuum). We therefore assume that the spin results obtained for each particle are completely determined by the values of that particle's hidden variables, which are set when the two particles are emitted by the source (or at least when they are close together). This assumption embodies the principle of **locality**, and the hidden variables carried by each particle are said to be **local hidden variables**.

This sketch summarises all we need to know about local hidden variable theories. Of course it leaves many questions unanswered. For example, why do spin components have only two possible values, and why do the hidden variables take random values within a probability distribution that gives precise agreement with quantum mechanics? We need not answer these questions. The important point is that we seem to have prised open a door that will allow a local hidden variable description of a Bohm-type experiment. But, when we examine the hidden-variable predictions more closely, you will see that this is an illusion.

1.3 Hidden variable predictions

A Bohm-type experiment involves spin measurements on many particle pairs. We shall consider just one pair, A and B, each with definite values of all their hidden variables, selected randomly from a suitable probability distribution.

Suppose, for the moment, that the hidden variables for particle A mean that it is bound to give a spin-up result for a spin measurement in the direction defined by α_1. Then particle A will always give spin up in that direction irrespective of the axis chosen for the spin measurement on particle B. This is unavoidable in any local hidden variables theory – Observer B could change his mind about which axis to use at the last moment, leaving no time for news of this decision to influence the hidden variables aboard Particle A. So, with its hidden variables set as assumed, Particle A will give spin-up for *both* of the axis combinations (α_1, β_1) and (α_1, β_2). With the alternative setting of its hidden variables, Particle A will give spin-down for *both* of the axis combinations (α_1, β_1) and (α_1, β_2). But no local hidden variable theory would predict that Particle A will

give spin up for the combination (α_1, β_1) and spin-down for the combination (α_1, β_2).

In any local hidden variables description of a Bohm-type experiment, the hidden-variable values of each particle determine with certainty whether it will give spin-up or spin-down along a chosen axis, irrespective of any spin measurements taken on the other particle.

Now let us suppose that the hidden variables of particles A and B are set in such a way that Observers A and B *agree* about the results for the axis combination (α_2, β_2). Then either (i) both observers get spin-up relative to their chosen axes or (ii) both observers get spin down. We consider first the case where both observers get spin-up results. Table 1.1 summarizes our knowledge about what would happen for all four axis combinations for this pair of particles:

Table 1.1 Predicted results of various spin measurements taken on a pair of particles whose hidden-variable values ensure that both particles would give spin up for the axis combination (α_2, β_2).

α_1	β_1	α_1	β_2	α_2	β_1	α_2	β_2
a	b	a	$+\hbar/2$	$+\hbar/2$	b	$+\hbar/2$	$+\hbar/2$

In this table, I have denoted a spin-up result by $+\hbar/2$. We know that this occurs for both of the measurements in the axis combination (α_2, β_2). From the boxed statement above, we can copy $+\hbar/2$ into all the other columns for α_2 and β_2. We do not know what the remaining results will be, but we do know that a spin measurement in the direction α_1 will give the same result in both the columns headed α_1 – we denote this unknown result by a. Similarly, a spin measurement in the direction β_1 will give the same result in both columns headed β_1, and we denote this unknown result by b.

Now look carefully at the table, remembering that the unknowns a and b can take either of the values $+\hbar/2$ or $-\hbar/2$. It is easy to see that no matter what choices you make for a and b, you cannot get disagreement in *all* of the combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1). For example, to get disagreement for the combinations (α_1, β_2) and (α_2, β_1), you will have to choose $a = b = -\hbar/2$, but this will give *agreement* for the combination (α_1, β_1).

We could go through a similar analysis for the case where both particles have spin-down in the combination (α_2, β_2). A similar table could be constructed, with all four $+\hbar/2$ entries replaced by $-\hbar/2$. We again find that we cannot have disagreement for *all* of the combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1). So our conclusion is:

In any local hidden variables description of a Bohm-type experiment, every set of hidden variable values that produce agreement for the axis combination (α_2, β_2) will also produce agreement for at least one of the combinations (α_1, β_1), (α_1, β_2) or (α_2, β_1).

This is where things start to unravel for local hidden variable theories.

Our hidden variable theory is designed to reproduce the predictions of quantum mechanics, and know what these are: agreement 85.4% of the time for the axis combination (α_2, β_2), but agreement only 14.6% of the time for each of the combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1). But we have seen that the hidden variable values that give agreement for the combination (α_2, β_2) must also give agreement for at least one of the combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1). It is easy to see that these percentages do not make any sense.

Suppose I say that, when I have lunch, I always eat vegetables, meat, fish or eggs. I also say that, whenever I eat vegetables, I also eat either meat or fish or eggs. Finally, I tell you that I eat vegetables for lunch 85.4% of the time, but I only eat meat for lunch 14.6% of the time, fish for lunch 14.6% of the time and eggs for lunch 14.6% of the time. What would your reaction be? You would probably say that I must be mistaken – the high proportion of lunches that contain vegetables is not consistent with the low proportion of lunches that contain either meat or fish or eggs. Do I ever eat vegetables by themselves? No, I insist, I always eat vegetables with either meat or fish or eggs. Of course, this is unbelievable – and so is the hidden-variables description of Bohm-type experiments.

In quantitative terms, suppose we accept that each of the axis combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1) gives agreement 14.6% of the time. Then we can say that *one or other* of these combinations gives agreement $14.6\% + 14.6\% + 14.6\% = 43.8\%$ of the time or less[1]. Our hidden variables model then predicts that the axis combination (α_2, β_2) can only give agreement 43.8% of the time or less. This clearly contradicts the quantum-mechanical prediction that this axis combination gives agreement 85.4% of the time.

One final point may need clarification. For any given pair of particles, our Bohm-type experiment uses just *one* of the four axis combinations. This means that we cannot directly see the pattern implied by Table 1.1 for individual particle pairs. However this does not affect our conclusions. In practice, we gather results using many particle pairs, with the axis combination chosen randomly for each pair. In the long run, this means that each set of hidden variable values will be explored many times, using all the different axis combinations. Provided there is no bias in the random choice of axis combinations, the predictions of a local variable theory will be tested fairly.

[1] I say "or less" because it is possible to get agreement for all of the combinations simultaneously. The estimate of 43.8% effectively triple-counts the rare case of total agreement, so the exact percentage is slightly less than 43.8%.

2. The CHSH inequality

2.1 Quantum mechanical predictions

The failure described above can be summarized as follows. For particles prepared in a singlet state, quantum mechanics predicts a relatively low probability of agreement for the axis combinations (α_1, β_1), (α_1, β_2) and (α_2, β_1) (where the angle between the two axes is 45°) but a relatively high probability of agreement for the axis combination (α_2, β_2) (where the angle between the two axes is 135°). We have seen that these probabilities cannot be explained by local hidden variables theories.

To investigate this clash more systematically, we follow Clauser, Horne, Shimony and Holt (CHSH) by introducing the *correlation function*

$$C(\theta_{\rm A}, \theta_{\rm B}) = P_{\rm agree}(\theta_{\rm A}, \theta_{\rm B}) - P_{\rm disagree}(\theta_{\rm A}, \theta_{\rm B}), \tag{2.1}$$

where $P_{\rm agree}(\theta_{\rm A}, \theta_{\rm B})$ is the probability of agreement between the two spin measurements when the axes are chosen at angles $\theta_{\rm A}$ and $\theta_{\rm B}$ and $P_{\rm disagree}(\theta_{\rm A}, \theta_{\rm B})$ is the probability of disagreement.

The correlation function is positive if agreement is more likely than disagreement, and it is negative if disagreement is more likely that agreement. The low probabilities associated with the (α_1, β_1), (α_1, β_2) and (α_2, β_1) give negative correlation functions, while the high probability associated with the (α_2, β_2) gives a positive correlation function.

To summarize the correlations for the four different axis combinations under study, we introduce the sum

$$\Sigma = C(\alpha_1, \beta_1) + C(\alpha_1, \beta_2) + C(\alpha_2, \beta_1) - C(\alpha_2, \beta_2). \tag{2.2}$$

We shall call Σ the *CHSH sum.*

For the axis combinations we have chosen, quantum mechanics predicts that each term in this sum is strongly negative (if we include the minus sign with the final term). We can therefore hope to confirm the clash between quantum mechanics and local hidden variable theories by making a quantum mechanical prediction for Σ and showing this is more negative than any value that can be achieved by a local hidden variables theory.

First, let's make the quantum mechanical prediction for a singlet state. For axes defined by the angles $\theta_{\rm A}$ and $\theta_{\rm B}$, we have seen that the probability of agreement is

$$P_{\rm agree}(\theta_{\rm A}, \theta_{\rm B}) = \sin^2\left[(\theta_{\rm B} - \theta_{\rm A})/2\right], \tag{Eqn 1.6}$$

so the probability of disagreement is

$$\begin{aligned} P_{\rm disagree}(\theta_{\rm A}, \theta_{\rm B}) &= 1 - P_{\rm agree}(\theta_{\rm A}, \theta_{\rm B}) \\ &= \cos^2\left[(\theta_{\rm B} - \theta_{\rm A})/2\right]. \end{aligned}$$

The correlation function is then

$$\begin{aligned} C(\theta_{\rm A}, \theta_{\rm B}) &= \sin^2\left[(\theta_{\rm B} - \theta_{\rm A})/2\right] - \cos^2\left[(\theta_{\rm B} - \theta_{\rm A})/2\right] \\ &= -\cos(\theta_{\rm B} - \theta_{\rm A}), \end{aligned} \tag{2.3}$$

where we have used the double angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ in the last step. Equation 2.3 is Equation 6.19 of Book 2.

With our choice of angles ($\alpha_1 = 0°$, $\alpha_2 = 90°$, $\beta_1 = 45°$, $\beta_2 = -45°$) we have

$$C(\alpha_1, \beta_1) = -\cos(45°) = -1/\sqrt{2}$$

$$C(\alpha_1, \beta_2) = -\cos(-45°) = -1/\sqrt{2}$$

$$C(\alpha_2, \beta_1) = -\cos(-45°) = -1/\sqrt{2}$$

$$C(\alpha_2, \beta_2) = -\cos(-135°) = +1/\sqrt{2}.$$

So

$$\Sigma = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\sqrt{2} = -2.828.$$

Now, it turns out that any local hidden variable theory is required to satisfy the CHSH inequality:

$$|\Sigma| \leq 2,$$

so the strongly-negative value of Σ predicted by quantum mechanics is incompatible with all local hidden variables theories. The CHSH inequality is quoted without proof in Chapter 6 of Book 2. A full proof is given in the next section.

2.2 A proof of the CHSH inequality

The CHSH inequality is an example of a Bell-type inequality for local hidden variable theories. Its proof does not involve any quantum mechanics – in fact, as we have just seen, quantum mechanical predictions violate the inequality and are therefore inconsistent with it.

In any local hidden variable theory, we have seen that a set of hidden-variable values that gives agreement for the axis combination (α_2, β_2) will also give agreement for at least one other axis combination. This is part of a more general result:

> In any local hidden variables description of a Bohm-type experiment, any given set of hidden-variable values produces agreement for an *even* number of the axis combinations (α_1, β_1), (α_1, β_2), (α_2, β_1) and (α_2, β_2). So, agreement may be found for none these combinations, two of them, or all four of them.

The first step in proving the CHSH inequality is to establish this result. We shall use the fact, noted earlier, that if a particle is set to give a definite result for a spin measurement along a given axis, it will always give the same result along that axis, no matter what choices are made for measuring the spin of the other particle. This allows us to fill in the following table where a, b, c and d can each take either of the values $+\hbar/2$

or $-\hbar/2$. Note that the variables a, b, c and d each appear twice because each axis occurs twice.

Table 2.1 Results of various spin measurements for a given pair of particles with a fixed set of local hidden variables. Each entry represents a result that would be obtained if a spin measurement along the specified axis were made. The spin components a, b, c and d can each take the values $+\hbar/2$ or $-\hbar/2$.

α_1	β_1	α_1	β_2	α_2	β_1	α_2	β_2
a	b	a	d	c	b	c	d

Since each spin measurement can give either $+\hbar/2$ or $-\hbar/2$, the product of the pair of results for a given axis combination is either equal to $+\hbar^2/4$ (for agreement) or to $-\hbar^2/4$ (for disagreement). There are four axis combinations, so if we multiply the $\pm\hbar^2/4$ factors together for all four axis combinations, we get

$$\text{product} = \begin{cases} +\dfrac{\hbar^8}{256} & \text{for an even number of agreements} \\ -\dfrac{\hbar^8}{256} & \text{for an odd number of agreements.} \end{cases} \tag{2.4}$$

However, multiplying the $\pm\hbar^2/4$ factors for the four axis combinations is equivalent to multiplying all the entries in the bottom row of Table 2.1. Because the variables a – d occur in pairs, and each has the value $\pm\hbar/2$, we get

$$\begin{aligned} \text{product} &= a^2 \times b^2 \times c^2 \times d^2 \\ &= \left(\pm\frac{\hbar}{2}\right)^2 \times \left(\pm\frac{\hbar}{2}\right)^2 \times \left(\pm\frac{\hbar}{2}\right)^2 \times \left(\pm\frac{\hbar}{2}\right)^2 \\ &= +\frac{\hbar^8}{256}. \end{aligned}$$

So, comparing with Equation 2.4, we see that there is always an *even* number of agreements – that is, either 0 or 2 or 4, but *not* 1 or 3.

Now consider pairs of particles emerging from the source with a given set of hidden-variable values (denoted here by V). For any given axis combination, characterized by the angles θ_A and θ_B, spin measurements on such a pair of particles will give either agreement or disagreement. Whichever result is obtained, it is obtained with *certainty*, for whenever we repeat a given measurement using a pair of particles with the same hidden-variable values, we must always get the same result.

Using many pairs of particles, all with the same set of hidden variable values, we can construct a set of restricted correlation functions $C_V(\alpha_1, \beta_1)$, $C_V(\alpha_1, \beta_2)$, $C_V(\alpha_2, \beta_1)$ and $C_V(\alpha_2, \beta_2)$, where the subscript reminds us that the set of hidden variable values is fixed to be V. For this set of hidden-variable values, each axis combination produces either agreement with certainty or disagreement with certainty, so each restricted correlation function is either equal to $+1$ (agreement with certainty) or is equal to -1 (disagreement with certainty).

Now we can use the restricted correlation functions to construct a restricted CHSH sum:

$$\Sigma_V = C_V(\alpha_1, \beta_1) + C_V(\alpha_1, \beta_2) + C_V(\alpha_2, \beta_1) - C_V(\alpha_2, \beta_2),$$

which again refers only to pairs of particles with the given set of hidden-variable values, V. We cannot say whether any given restricted correlation function will take the value $+1$ or -1, but we do know that there must be an even number of agreements and therefore an even number of $+1$ values. This severely limits the values that Σ_V can have, as we shall now show. The number of agreements is either 0 or 4 or 2. We shall consider each of these these cases in turn.

0 agreements In this case, each of the restricted correlation functions is equal to -1 so the restricted CHSH sum is

$$\Sigma_V = -1 - 1 - 1 + 1 = -2.$$

4 agreements In this case, each of the restricted correlation functions is equal to $+1$ so the restricted CHSH sum is

$$\Sigma_V = +1 + 1 + 1 - 1 = +2.$$

2 agreements In this case, two of the restricted correlation functions are equal to $+1$ and the other two are equal to -1. Table 2.2 shows the possible ways these values can be combined in the restricted CHSH sum.

Table 2.2 The possible ways in which restricted correlation functions with 2 agreements and 2 disagreements can be combined to produce the restricted CHSH sum Σ_V.

$C_V(\alpha_1, \beta_1)$	$C_V(\alpha_1, \beta_2)$	$C_V(\alpha_2, \beta_1)$	$C_V(\alpha_2, \beta_2)$	Σ_V
$+1$	$+1$	-1	-1	$+2$
$+1$	-1	$+1$	-1	$+2$
-1	$+1$	$+1$	-1	$+2$
$+1$	-1	-1	$+1$	-2
-1	$+1$	-1	$+1$	-2
-1	-1	$+1$	$+1$	-2

Note that, in all cases, Σ_V is equal to $+2$ or -2. This is a direct consequence of the fact that the number of agreements must be an even number. With *one* agreement we could have $\Sigma_V = -1 - 1 - 1 - 1 = -4$, and with *three* agreements we could have $\Sigma_V = 1 + 1 + 1 + 1 = 4$, but these values are ruled out by our previous argument. Drawing all these results together, we see that:

Possible values of restricted CHSH sums

Within any local hidden variable theory, the only possible values for Σ_V are $+2$ and -2.

So far, we have only considered particle pairs with a fixed set of hidden-variable values, V. To get the final CHSH inequality, we relax this restriction and consider *all* particle pairs emitted by the source. The CHSH sum Σ is then obtained by averaging Σ_V over all the allowed sets of hidden variables, which occur according to some unknown probability distribution.

The key point is that, for each set of hidden-variable values, Σ_V is equal to $+2$ or -2 so, no matter how the hidden variables are distributed, the effect of averaging can never take Σ outside the range from -2 to $+2$. The value $\Sigma = -2$ would occur if Σ_V were equal to -2 for all possible hidden-variable values, while the value $\Sigma = +2$ would occur if Σ_V were equal to $+2$ for all possible hidden-variable values; beyond these extreme cases, we have $-2 < \Sigma < +2$. We have therefore proved:

The CHSH inequality

Within any local hidden variables theory,

$$|\Sigma| \leq 2.$$

Remember that quantum mechanics predicts values of $|\Sigma|$ that lie well beyond the CHSH limit of 2, and these predictions have been amply confirmed by experiments (e.g. those by Aspect). Local hidden variable theories cannot recover from this.

2.3 Where do hidden variables go wrong?

Much of our effort has been gone into demonstrating that local hidden variable theories cannot reproduce the trusted predictions of quantum mechanics. It is tempting to see this as a paradox, and to scour through the arguments for loopholes or logical flaws. Diehard opponents of quantum mechanics might well undertake this frustrating search, but I don't recommend it. From the viewpoint of quantum mechanics, there is no paradox: it is not the details of the hidden variables theories that are shaky, but their entire foundation.

A local hidden variable theory pictures each particle as emerging from the source with definite hidden-variable values. This means that each particle is predestined to give a definite result (either $+\hbar/2$ or $-\hbar/2$, depending on the hidden-variable values) for the measurement of any chosen spin component. We then assume that any particle that is set to give $+\hbar/2$ along a particular axis will give that value no matter which axis is chosen for spin measurements on the other particle. This creates a characteristic pattern linking the results for different axis combinations (Table 2.1), and we have seen that this leads to a clash with the predictions of quantum mechanics and the results of experiments.

Quantum mechanics brushes all this aside by giving a totally different description. In quantum mechanics, the singlet state is an entangled spin state. This prevents us from thinking about the spin of Particle A or the

spin of Particle B separately. The singlet state can be written as

$$|\text{singlet}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_{\mathbf{n}}\downarrow_{\mathbf{n}}\rangle - |\downarrow_{\mathbf{n}}\uparrow_{\mathbf{n}}\rangle\right), \tag{2.5}$$

where $\mathbf{n}$ is an arbitrary direction. This shows that the spin components of both particles in the $\mathbf{n}$-direction will give one $+\hbar/2$ result and one $-\hbar/2$ result but, prior to any measurement, we cannot say *which* particle will give *which* of these values.

In general, quantum mechanics predicts *probabilities* rather than certainties. The quantum-mechanical calculation for a singlet state was given in Section 1.1. You saw that the probability of getting spin-up results for both particles is

$$\text{probability} = |\langle \uparrow_{\theta_{\mathrm{A}}}\uparrow_{\theta_{\mathrm{B}}} |\text{singlet}\,\rangle|^2 = \frac{1}{2}\sin^2\left[(\theta_{\mathrm{B}} - \theta_{\mathrm{A}})/2\right], \quad \text{(Eqn 1.4)}$$

where θ_{A} and θ_{B} are the angles in Figure 1.1. Because the singlet state is entangled, the probability that emerges is *not* the product of a function of θ_{A} and a function θ_{B}. Instead, there are correlations between the two spin results. For example, if both spins are measured along the same axis, Equation 1.4 gives a probability of zero – exactly as expected for a singlet state, where the two spins are opposite. However, Equation 1.4 shows that probabilistic correlations exist at other angles, as well.

Correlations can also be built into hidden variable models. For example we can insist that a particle that is certain to give spin-up in a given direction will be accompanied by a particle that is certain to give spin-down in the same direction. But, in a local hidden variable theory, all such correlations are determined when the particles are close together; when the particles are far apart, the principle of locality implies that any last-minute changes in θ_{B} cannot influence the results for particle A. This restricts the correlations that are possible in a local hidden variable theory.

Our detailed arguments have shown that the quantum correlations are too strong to be explained classically. Since experiments confirm the quantum-mechanical correlations, the point is not that hidden variable theories are logically flawed, but that they are wishful thinking. They do not describe the world we live in.

Appendix: Other Bohm-type experiments

The Bohm-type experiment discussed in these notes can be generalized in different ways. For the sake of completeness, this appendix briefly outlines some of these generalizations.

1. Axes oriented in different directions

We have considered the special case $\alpha_1 = 0°$, $\alpha_2 = 90°$, $\beta_1 = 45°$ and $\beta_2 = -45°$. However, the CHSH inequality does not depend on this choice. We can therefore compare quantum mechanical predictions with the CHSH inequality for other sets of axes. Figure 3.1 shows the case where α_1 and α_2 are orthogonal, and β_1 and β_2 are orthogonal. The β axes are rotated with respect to the α axes by θ, but θ need no longer be equal to 45°. We then have:

$$\alpha_1 = 0°, \quad \alpha_2 = 90°, \quad \beta_1 = \theta \quad \text{and} \quad \beta_2 = \theta - 90°.$$

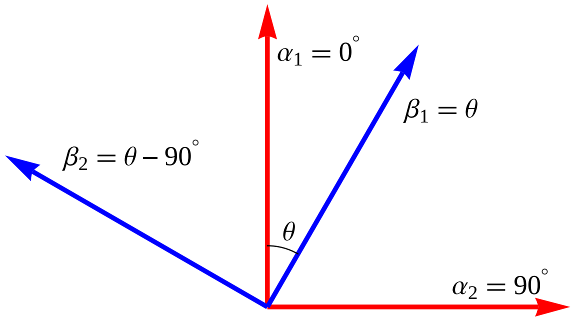

Figure 3.1 A more general choice of axes for a Bohm-type experiment.

We continue to consider two spin-$\frac{1}{2}$ particles in a singlet state, so the correlation function for angles θ_A and θ_B is still given by

$$C(\theta_\mathrm{A}, \theta_\mathrm{B}) = -\cos(\theta_\mathrm{B} - \theta_\mathrm{A}) \qquad \text{(Eqn 2.3)}$$

and the CHSH sum becomes

$$\begin{aligned}\Sigma &= C(\alpha_1, \beta_1) + C(\alpha_1, \beta_2) + C(\alpha_2, \beta_1) - C(\alpha_2, \beta_2) \\ &= -\cos(\theta) - \cos(\theta - 90°) - \cos(\theta - 90°) + \cos(\theta - 180°) \\ &= -\cos\theta - \sin\theta - \sin\theta - \cos\theta \\ &= -2(\cos\theta + \sin\theta). \qquad (3.1)\end{aligned}$$

This agrees with our previous prediction $\Sigma = -2\sqrt{2}$ for the special case $\theta = 45°$. The quantum-mechanical prediction of Equation 3.1 is compared with the CHSH inequality in Figure 3.2. The shaded regions show where

the CHSH inequality is exceeded, and it is here that local-hidden variable theories fail. For example, they fail at all angles $0° < \theta < 90°$, with the largest deviation occurring at $\theta = 45°$.

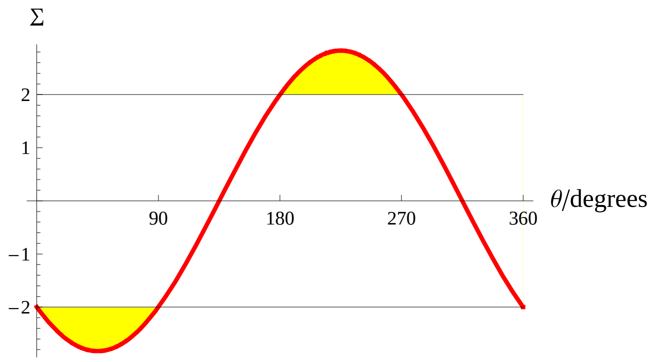

Figure 3.2 Comparing the quantum-mechanical predictions of Equation 3.1 with the CHSH inequality.

Another way of generalizing the choice of axes is shown in Figure 3.3. Here, we use four generally non-orthogonal axes, each separated from the next by the same angle, θ. This is the situation explored in the *Ghostly action at a distance* archive video segment mentioned in Chapter 6 of Book 2.

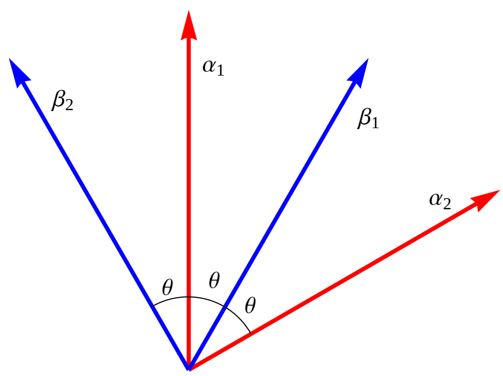

Figure 3.3 Another choice of axes for a Bohm-type experiment.

In this case, for spin-$\frac{1}{2}$ particles in a singlet state, three of the correlation functions are equal to $-\cos\theta$ and the other is equal to $-\cos(3\theta)$, so the quantum-mechanical prediction for the CHSH sum is

$$\Sigma = \cos(3\theta) - 3\cos\theta. \tag{3.2}$$

This quantum-mechanical prediction is compared with the CHSH inequality in Figure 3.4. The shaded regions show where the CHSH inequality is exceeded, and it is here that local-hidden variable theories fail. For example, they fail at all angles $0 < \theta < 68.5°$, with the largest deviation occurring at $\theta = 45°$.

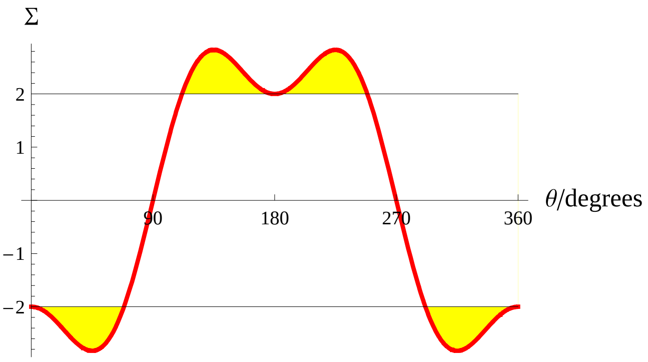

Figure 3.4 Comparing the quantum-mechanical predictions of Equation 3.2 with the CHSH inequality.

2. Entangled polarization states of photon pairs

From a practical point of view, it is more convenient to carry out experiments with photons rather than spin-$\frac{1}{2}$ particles. Corresponding to the singlet state for two spins, we can consider the entangled two-photon polarization state

$$|\mathrm{A}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{VH}\rangle - |\mathrm{HV}\rangle\right), \tag{3.3}$$

This state is discussed in Section 7.2.3 of Book 2.

where $|\mathrm{V}\rangle$ and $|\mathrm{H}\rangle$ represent states of vertical and horizontal polarization relative to the z-axis. More generally, states of vertical and horizontal polarization relative to an axis at the angle θ in the xz-plane are denoted by $|\mathrm{V}_\theta\rangle$ and $|\mathrm{H}_\theta\rangle$, and the state $|A\rangle$ can equally-well be written in terms of them:

$$|\mathrm{A}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{V}_\theta\mathrm{H}_\theta\rangle - |\mathrm{H}_\theta\mathrm{V}_\theta\rangle\right). \tag{3.4}$$

Our previous calculations for a singlet state of two spin-$\frac{1}{2}$ particles, can be repeated for photons in the state $|A\rangle$. There is one major difference: the angles in the formulae for photons are generally double the angles in the formulae for spin-$\frac{1}{2}$ particles. This correspondence is shown in Table 3.1.

Table 3.1 Comparison between calculations for a pair of spin-$\frac{1}{2}$ particles in a singlet state and a pair of photons in the state $|\mathrm{A}\rangle$ of Equation 3.3.

spin-$\frac{1}{2}$ particles	photons
$\vert\mathrm{singlet}\rangle = \frac{1}{\sqrt{2}}\left(\vert\uparrow\downarrow\rangle - \vert\downarrow\uparrow\rangle\right)$	$\vert\mathrm{A}\rangle = \frac{1}{\sqrt{2}}\left(\vert\mathrm{VH}\rangle - \vert\mathrm{HV}\rangle\right)$
$\vert\uparrow_\theta\rangle = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{bmatrix}$	$\vert\mathrm{V}_\theta\rangle = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$
$\vert\downarrow_\theta\rangle = \begin{bmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}$	$\vert\mathrm{H}_\theta\rangle = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$
$P_{\mathrm{agree}} = \vert\langle\uparrow_\theta\uparrow_\theta\vert\mathrm{singlet}\rangle\vert^2$	$P_{\mathrm{agree}} = \vert\langle\mathrm{V}_\theta\mathrm{V}_\theta\vert\mathrm{A}\rangle\vert^2$
$P_{\mathrm{agree}} = \sin^2\left[(\theta_\mathrm{B}-\theta_\mathrm{A})/2\right]$	$P_{\mathrm{agree}} = \sin^2(\theta_\mathrm{B}-\theta_\mathrm{A})$
$C(\theta_\mathrm{A},\theta_\mathrm{B}) = -\cos(\theta_\mathrm{B}-\theta_\mathrm{A})$	$C(\theta_\mathrm{A},\theta_\mathrm{B}) = -\cos\left[2(\theta_\mathrm{B}-\theta_\mathrm{A})\right]$

Using this correspondence, and comparing with Equation 3.1, we see that pairs of photons in the polarization state $|\mathrm{A}\rangle$, analyzed with axes at angles $\alpha_1 = 0°$, $\alpha_2 = 90°$, $\beta_1 = \theta$ and $\beta_2 = \theta - 90°$, give

$$\Sigma = -2\big(\cos(2\theta) + \sin(2\theta)\big). \tag{3.5}$$

The comparison between this quantum-mechanical prediction and the CHSH inequality is shown in Figure 3.5. For photons, the largest violation occurs at $\theta = 22.5°$.

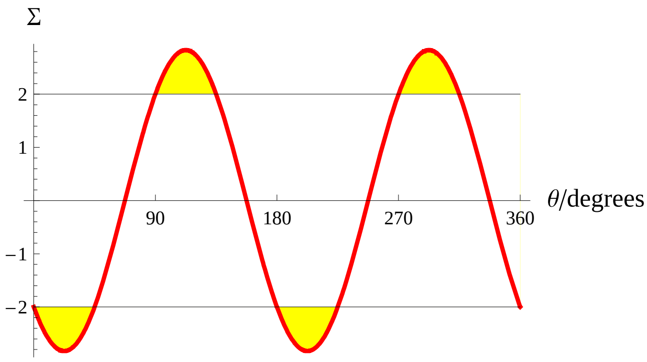

Figure 3.5 Comparing the quantum-mechanical predictions of Equation 3.5 with the CHSH inequality.

3. Other entangled states

It is also possible to conduct Bohm-type experiments with other entangled states. For example, the Aspect experiments used the photon pairs in the state

$$|\mathrm{B}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{VV}\rangle + |\mathrm{HH}\rangle\right). \tag{3.6}$$

This state also violates the CHSH inequality, as shown in Section 6.4.2 of Book 2.